CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar, Friday conversations:,
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: colimits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimers.

Disclaimer I.– The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

Disclaimer II.– Sometimes, absolute rigour is not followed[a]. In its place, from time to time, we will seek to give the reader an intuitive feel for what the concepts of category theory are and how they relate to our ongoing research within CIP, CAP and CCRT.

[a] All is assumed to be subsequently clarified on request though.

Disclaimer III.– The reader will find repetitions and reprises from the preceding CCRT[n], they correspond to some points which were skipped or uncompletely treated during preceding seminars.

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Matters

1. Representation theory (or theories)
   1. Geometric point of view
   2. Combinatorial point of view (Ram and Barcelo manifesto)
   3. Categorical point of view
2. From groups to algebras
   Here is a bit of rep. theory of the symmetric group, deformations, idempotents
3. Irreducible and indecomposable modules
4. Characters, central functions and shifts.
   Here are (some of) Lascoux and Schützenberger's results
5. Reductibility and invariant inner products
   Here stands Joseph's result
6. Commutative characters
   Here are time-ordered exponentials, iterated integrals, evolution equations and Minh's results
7. Lie groups Cartan theorem
   Here is BTT

CCRT[31] Categorical aspects of Lazard's elimination theorems.

Plan of this talk.

1. Categories of this talk: many but mainly Lie algebras and groups.
2. Universal problems
   1. wrt a functor
   2. as an initial element (indeed an arrow) of a "derived category"
3. Variations about semidirect products.
   1. Groups
   2. Lie algebras
4. Link with Lazard's elimination
5. Concluding remarks

Outline

Short Exact Sequences (SES)

1. Let **k** be a commutative ring. Take one of your favorite category "over" **k** − **Mod** (abelian groups, **k**-modules, algebras **k** − **Alg**, Lie algebras **k** − **Lie**, non-unital associative algebras **k** − **AA**, associative algebras with unit **k** − **AAU**, superalgebras).
2. A short exact sequence (SES) is of the form

$$0 \longrightarrow \mathcal{A}_1 \xrightarrow{\alpha_{31}} \mathcal{A}_3 \xrightarrow{\alpha_{23}} \mathcal{A}_2 \longrightarrow 0$$

Figure: Fig.1. A SES.

3. The (SES) form a category. Morphisms are triples $(\varphi_1, \varphi_3, \varphi_2)$ such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_1 \arrow[r, "\alpha_{31}"] \arrow[d, "\varphi_1"] & \mathcal{A}_3 \arrow[r, "\alpha_{23}"] \arrow[d, "\varphi_3"] & \mathcal{A}_2 \arrow[r] \arrow[d, "\varphi_2"] & 0 \\
0 \arrow[r] & \mathcal{B}_1 \arrow[r, "\beta_{31}"] & \mathcal{B}_3 \arrow[r, "\beta_{23}"] & \mathcal{B}_2 \arrow[r] & 0
\end{tikzcd}$$

Short Exact Sequences (SES)/2

4. The prototype of a short exact sequence (SES) is of the form

$$0 \longrightarrow \mathcal{J} \xrightarrow{j} \mathcal{A} \xrightarrow{s} \mathcal{A}/\mathcal{J} \longrightarrow 0$$

5. And a SES as 1 being given, one can compute a prototype isomorphic to Fig.1 with $\mathcal{J} = Im(\alpha_{31}) = Ker(\alpha_{12})$.
6. Now, taking **k**-Lie algebras, $\mathfrak{g}_i,\ i = 1 \cdots 3$, let us remark that, saying $\mathfrak{g}_3 = \mathfrak{g}_1 \rtimes \mathfrak{g}_2$ amounts to say that the SES of Lie algebras

$$0 \longrightarrow \mathfrak{g}_1 \longrightarrow \mathfrak{g}_3 \xrightarrow{\alpha_{23}} \mathfrak{g}_2 \longrightarrow 0$$

is split (i.e. α_{23} admits a section σ). Then, it reads

$$0 \longrightarrow \mathfrak{g}_1 \longrightarrow \mathfrak{g}_3 \underset{\alpha_{23}}{\overset{\sigma}{\leftrightarrows}} \mathfrak{g}_2 \longrightarrow 0$$

and, in fact, $\mathfrak{g}_3 = ker(\alpha_{23}) \rtimes Im(\sigma)$.

SES and gradings.

7. Such a (complemented) nesting amounts to have a $\mathbb{B}$-grading. Where $(\mathbb{B}, +)$ is the additive part of the boolean semiring whose law reads

$$\begin{array}{c|c|c} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 1 \end{array} \tag{1}$$

8. Indeed, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$, we can set $\mathfrak{g}_0 = \mathfrak{b}$ and $\mathfrak{g}_1 = \mathfrak{h}$ and check that, in this way, $\mathfrak{g}$ is $\mathbb{B}$-graded.
9. Let us look closer to Bourbaki's general treatise and see how this was already implemented in classical literature.

Generalized gradings (or gradings in the large).

10. In [2] ch. II §11.1, def 1, the set of degrees (of a graded module of commutative group) is only a set and later a monoid (when we come to rings). This is followed by [34]. For a non-unital algebra (like a Lie algebra), we can give the following definition

Definition

Let $\mathcal{A}$ be a **k**-algebra (can be non-associative and non-unital) and S a semigroup. A structure of S-graded **k**-algebra on $\mathcal{A}$ is the data of

1. A decomposition
$$\mathcal{A} = \oplus_{s \in S} \mathcal{A}_s \tag{2}$$
2. such that
$$\mathcal{A}_s.\mathcal{A}_t \subset \mathcal{A}_{st} \qquad \forall s,t \in S \tag{3}$$

11. The notions of graded subalgebra, graded ideal and graded quotient and their properties are similar as for the classical case (see [2] "Graded modules", ch. II §11.3 Propositions 2-3 and their Corollaries do this as exercises).

Example: words and the Free Lie algebra.

1. With $\mathcal{X} = \{a, b\}$ and the bigading by $w \mapsto (|w|_a, |w|_b)$.

$(\lvert w\rvert_a, \lvert w\rvert_b)$	words
$(3,0)$	a^3
$(2,1)$	$a^2b,\ aba,\ ba^2$
$(1,2)$	$ab^2,\ bab,\ b^2a$
$(0,3)$	b^3
$\vdots$	$\vdots$
(p,q)	$a^p \sqcup\!\sqcup\, b^q$

2. With $\mathcal{X} = \{a, b\}$ and the grading by $w \mapsto d(w) = 2.|w|_a + |w|_b$ we get.

$d(w)$	words
0	$1_{\mathcal{X}^*}$
1	b
2	$a,\ b^2$
3	$ab,\ ba,\ b^3$
4	$a^2,\ ab^2,\ bab,\ b^2a,\ b^4$

Words

We recall basic definitions and properties of the free monoid [1]:

- An alphabet is a set $\mathcal{X}$ (of variables, indeterminates, generators etc.)
- Words of length n (set X^n) are mappings $w : [1 \cdots n] \to X$. The letter at place j is $w[j]$, the empty word 1_{X^*} is the sole mapping $\emptyset \to X$ (i.e. of length 0). As such, we get actions, by composition
  - of $\mathfrak{S}_n$ on the right (noted $w.\sigma$) and
  - of the transformation monoid X^X on the left
- Words concatenate by shifting and union of domains, this law is noted *conc*
- $(X^*, conc, 1_{X^*})$ is the free monoid of base X.
- Given a total order on X, (X^*) is totally ordered by the graded lexicographic ordering $\prec_{glex}$ (length first and then lexicographic from left to right). This ordering is compatible with the monoid structure.

[1] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997)

An example: Lazard elimination.

12. Let X be a set and $\mathbf{k}$ a commutative ring. It can be shown that $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \subset \mathbf{k}\langle X\rangle$. With $S = \mathbb{N}^{(X)}$ $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ is a S-graded submodule of $\mathbf{k}\langle X\rangle$.
13. With a partition $X = B \sqcup Z$, we get the partial degrees for the words

$$|w|_Z = \sum_{z\in Z} |w|_z \text{ and } |w|_B = \sum_{b\in B} |w|_b \tag{4}$$

14. Considering the $\mathbb{N}^{(X)}$-grading given by the fine grading given by

$$w \mapsto (w)_X = (|w|_x)_{x\in X} \in \mathbb{N}^{(X)} \tag{5}$$

we have a fine grading of $\mathbf{k}\langle X\rangle$ given by the submodules $(\alpha \in \mathbb{N}^{(X)})$

$$\mathbf{k}_\alpha\langle X\rangle = span_{\mathbf{k}}\{w\}_{(w)_X=\alpha} \tag{6}$$

An example: Lazard elimination/2.

15. Because of the partition of the basis, we have

$$\mathbf{k}\langle X\rangle = \oplus_{\alpha\in\mathbb{N}^{(X)}}\mathbf{k}_\alpha\langle X\rangle \tag{7}$$

and this is a $\mathbb{N}^{(X)}$-grading (fine grading).

16. Then

$$\mathcal{J}_{pol} = \oplus_{|\alpha|_Z\geq 1}\mathbf{k}_\alpha\langle X\rangle \text{ and } \mathcal{J}_{Lie} = \mathcal{J}_{pol}\bigcap\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \tag{8}$$

are ideals respectively of $\mathbf{k}\langle X\rangle$ and $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$.

17. We have

$$0 \longrightarrow \mathcal{J}_{Lie} \xrightarrow{\ j\ } \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \xrightarrow{\ s\ } \mathcal{L}ie_{\mathbf{k}}\langle X\rangle\Big/\mathcal{J}_{Lie} \longrightarrow 0$$

18. Is is easy to see that, due to (7), SES of 17 is split i.e.

$$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle = \mathcal{J}_{Lie}\oplus\mathcal{L}ie_{\mathbf{k}}\langle B\rangle \tag{9}$$

Lazard elimination theorem (LET).

19 **Theorem A.–** In the preceding conditions.
i) The ideal $\mathcal{J}_{Lie}$ is a free Lie algebra, with basic family

$$C_B(Z) = \{ad_w(z)\}_{\substack{w\in B^* \\ z\in Z}} \tag{10}$$

where, if $w = x_1 \cdots x_n$, $ad_w \in End(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle)$ is the composition

$$ad_{x_1} \circ \cdots \circ ad_{x_n} \tag{11}$$

(monoidal adjoint representation).
ii) The map $u:\ \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \to \mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ such that $u(wz) = ad_w(z)$ and $u(b) = b$ is an isomorphism of **k**-Lie algebras

$$u : \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \to \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \tag{12}$$

Proof (and towards semidirect products)

20. We now describe the action of $\mathcal{L}ie_{\mathbf{k}}\langle B\rangle$ on $\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle$ by derivations and thus form the semidirect product

$$\mathfrak{g} = \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \tag{14}$$

such that, for all $b \in B$, this element acts on $\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle$ by $\alpha(b, wz) = bwz$ which is uniquely extended as a derivation of $\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle$ (see [3] Ch II §2.8 Corollary of Proposition 8).

21. Then, we have an arrow $u: \ \mathfrak{g} \to \mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ such that:
$$u(wz) = ad_w(z) \text{ and } u(b) = b$$

22. But, as the target is free, we can buid $v: \ \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \to \mathfrak{g}$ by $v(x) = x$ for all $x \in X = B \sqcup Z$.

23. It is now routine to check that u, v are mutually inverse.

Why is LET universal over semidirect products ?

20. The preceding proof provides us a (split) SES

$$0 \longrightarrow \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \longrightarrow \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \xrightarrow{\alpha_{23}} \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \longrightarrow 0$$

21. Let us now consider a situation $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$ where $\mathfrak{h}$ is an ideal and $\mathfrak{b}$ is a Lie subalgebra (hence we have the semidirect product $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$).
22. Let B (resp. Z) be a generating subset of $\mathfrak{b}$ (resp. $\mathfrak{h}$) *as Lie subalgebra* (resp. *as ideal*).
23. One can check that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \arrow[r, "\alpha_{31}"] \arrow[d, "\varphi_1"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "\alpha_{23}"] \arrow[d, "\varphi_3"] & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \arrow[r] \arrow[d, "\varphi_2"] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r, "\beta_{31}"] & \mathfrak{g} \arrow[r, "\beta_{23}"] & \mathfrak{b} \arrow[r] & 0 \end{tikzcd}$$

The arrows α_{ij} being constructed after u and v, the φ_j and the β_{ij} accordingly.

Semidirect products as colimits ?

24 As regards great generality, we have a precious indication in MO [41] where the question is about Bourbaki (General Topology III, §2.10 Prop. 27) characterization:
Proposition A.– Let $h \mapsto \phi_h$ a morphism of groups $H \to Aut(N)$ and $f\colon N \to G$, $g\colon H \to G$ be two homomorphisms into a group G, such that

$$f(\phi_h(n)) = g(h)f(n)g(h^{-1}) \tag{15}$$

for all $n \in N$, $h \in H$. Then there is a unique homomorphism $k\colon N \rtimes H \to G$ extending f and g in the usual sense.

25 The MO remains unsatisfied because the condition $f(\phi_h(n)) = g(h)f(n)g(h^{-1})$ is a condition on elements of groups, rather than a condition that says that some diagram is commutative. His question is: are semi-direct products in the category of groups categorical limits ?

Semidirect products as colimits ?/2

26. By far, the most general (but encapsulated) answer is that of Andreas Thom.
Let $\mathrm{Mor}(Gp)$ be the category whose objects are homomorphisms of groups and morphisms are commutative diagrams. Let C be the category of "groups acting on groups" whose objects are pairs of groups (G_2, G_1) together with a homomorphism $G_2 \to \mathrm{Aut}(G_1)$. Morphisms in this category are equivariant homomorphisms.
Now, there is a natural forgetful functor $T : \mathrm{Mor}(Gp) \to C$ which sends $G_2 \to G_1$ to the pair (G_2, G_1) with the homomorphism $G_2 \to \mathrm{Aut}(G_1)$ given by conjugation. Now, almost by definition, the crossed product is the left-adjoint of this forgetful functor. Indeed, the left adjoint is easily seen to map (G_2, G_1) with $G_2 \to \mathrm{Aut}(G_1)$ to the inclusion $G_2 \to G_1 \rtimes G_2$.
Being a left-adjoint, the "crossed product" maps colimits to colimits.

Two variations on (categorical) semidirects products: groups/1.

27 Let us see what (15) means in terms of commutative diagrams.

$$\begin{tikzcd}
G_2 \times G_1 \arrow[r, "f_2 \times f_1"] \arrow[d, "\alpha"] & G \times G \arrow[d, "Ad_G"] \\
G_1 \arrow[r, "f_1"] & G
\end{tikzcd}$$

Figure: Equivariance in **Grp**.

28 The category in question is as follows
Objects.– Pairs (f_1, f_2) from groups G_i, $i = 1, 2$ to a group G satifying (15)
Morphisms.– Elements of $Hom_{\mathbf{Grp}}(G, H)$ as it can be proved that, if (f_1, f_2) is admissible as object and $\varphi \in Hom_{\mathbf{Grp}}(G, H)$, then $(\varphi \circ f_1, \varphi \circ f_2)$ is admissible as object.

Two variations on (categorical) semidirects products: groups/2.

29. Transitivity of objects.

$$\begin{array}{ccccc} G_2 \times G_1 & \xrightarrow{f_2 \times f_1} & G \times G & \xrightarrow{\varphi \times \varphi} & H \times H \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle Ad_G} & & \downarrow{\scriptstyle Ad_H} \\ G_1 & \xrightarrow{f_1} & G & \xrightarrow{\varphi} & H \end{array}$$

Figure: Transitivity of objects by morphisms.

30. The category in question is as follows
Objects.– Pairs (f_1, f_2) from groups $G_i,\ i = 1, 2$ to a group G satifying (15)
Morphisms.– Elements of $Hom_{\mathbf{Grp}}(G, H)$ as it can be proved that, if (f_1, f_2) is admissible as object and $\varphi \in Hom_{\mathbf{Grp}}(G, H)$, then $(\varphi \circ f_1, \varphi \circ f_2)$ is admissible as object.

31. It can be shown that **Proposition A** says that $G_1 \rtimes G_2$ with its pair (j_1, j_2) is a initial element of this category.

Two variations on (categorical) semidirects products: **k**-Lie algebras.

32. For **k**-Lie algebras, the same holds in terms of commutative diagrams with transitivity and we can state Proposition B.

$$\begin{tikzcd} \mathfrak{g}_2 \times \mathfrak{g}_1 \arrow[r, "f_2\times f_1"] \arrow[d, "\alpha"] & \mathfrak{g} \times \mathfrak{g} \arrow[r, "\varphi\times\varphi"] \arrow[d, "ad^{\mathfrak{g}}"] & \mathfrak{h} \times \mathfrak{h} \arrow[d, "ad^{\mathfrak{h}}"] \\ \mathfrak{g}_1 \arrow[r, "f_1"] & \mathfrak{g} \arrow[r, "\varphi"] & \mathfrak{h} \end{tikzcd}$$

33. **Proposition B.–** Let $a \mapsto \alpha(a, \bullet)$ a morphism of **k**-Lie algebras $\mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$ and $f_1 \colon \mathfrak{g}_1 \to \mathfrak{g}$, $f_2 \colon \mathfrak{g}_2 \to \mathfrak{g}$ be two homomorphisms of **k**-Lie algebras to $\mathfrak{g}$, such that (15) i.e.

$$f_1(\alpha(a, b)) = [f_2(a), f_1(b)] \tag{16}$$

for all $a \in \mathfrak{g}_2$, $b \in \mathfrak{g}_1$ is fulfilled. Then there is a unique homomorphism $k \colon \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \to \mathfrak{g}$ extending f_1 and f_2 in the usual sense.

Relations with generators/1

34 We have the following
Proposition C.– The data being that of Proposition B, let Z (resp. B) be a set of generators of $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$).In order that (16) be true for all $a \in \mathfrak{g}_2$, $b \in \mathfrak{g}_1$ it is sufficient that it be true for $(a, b) \in Z \times B$.

35 **Proof.–** So, we suppose that (15) holds for all $(a, b) \in Z \times B$. Firstly, we fix $a \in Z$ and remark that

$$D_1 : \ x \mapsto f_1(\alpha(a, x)) \text{ and } D_2 : \ x \mapsto [f_2(a)f_1(x)] \tag{17}$$

are f_1-primitive i.e. $D_i : \ \mathfrak{g}_1 \to \mathfrak{g}$ satisty

$$D([x, y]) = [D(x), f_1(y)] + [f_1(x), D(y)] \tag{18}$$

one checks easily that (i) $D_1 - D_2$ fulfils the same identity and (ii) $\ker(D_1 - D_2)$ is a Lie subalgebra of $\mathfrak{g}_1$. This proves that for all $a \in \mathfrak{g}_2$ and $P \in \mathfrak{g}_1$, (15) is fulfilled.

A remark

36. Elements $D \in Hom_{\mathbb{Z}} A, B$ where $A, B \in \mathbf{k} - \mathbf{Alg}$ are arbitrary algebras (not necessarily assciative nor unital) which fulfill (18) are called f-primitive because this identity is equivalent to

$$\begin{tikzcd} A \otimes A \arrow[r, "D \otimes f + f \otimes D"] \arrow[d, "\mu_A"] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "D"] & B \end{tikzcd}$$

37. Label $D \otimes f + f \otimes D$ is reminiscent of "elements u-primitive" in [3] Ch II §1.1 definition 1 with $u = f$.
38. These elements form a subgroup of $Hom_{\mathbb{Z}}(A, B)$ and their kernel is always a subalgebra of A.

Relations with generators/2

39 We now remark that the fact that $a \mapsto \alpha(a, \bullet)$ is a Lie morphism amounts to saying that, for all $P \in \mathfrak{g}_1,\ a, b \in \mathfrak{g}_2$

$$\alpha(a, \alpha(b, P)) - \alpha(b, \alpha(a, P)) = \alpha([a, b], P)$$

40 Then, for all $a, b \in \mathfrak{g}_2$ such that (15) is fulfilled for all $P \in \mathfrak{g}_1$, we have

$$\begin{aligned} &f_1(\alpha([a, b], P)) = f_1(\alpha(a, \alpha(b, P))) - f_1(\alpha(b, \alpha(a, P))) = \\ &[f_2(a), f_1(\alpha(b, P))] - [f_2(b), f_1(\alpha(a, P))] = \\ &[f_2(a), [f_2(b), f_1(P)]] - [f_2(b), [f_2(a), f_1(P)]] = \\ &[[f_2(a), f_2(b), f_1(P)] \end{aligned}$$

which shows that the set

$$\mathfrak{H} = \{a \in \mathfrak{g}_2 \mid (\forall P \in \mathfrak{g}_1)(f_1(\alpha(a, P)) = [f_2(a), f_1(P)])\} \qquad (19)$$

is a Lie subalgebra of $\mathfrak{g}_2$. As $B \subset \mathfrak{H}$, this entails that $\mathfrak{H} = \mathfrak{g}_2$ which proves the claim.

Relations with generators/3

Presentations with commutations

41. Let $\mathbf{k}$ be a ground ring and $\mathfrak{g}$ a Lie algebra given by generators and relations[a]

$$\mathfrak{g} = \langle\ X\ ;\ \mathbf{R}\ \rangle_{\mathbf{k}-\mathbf{Lie}} \tag{20}$$

$\mathfrak{g}$ can be constructed as

$$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \Big/ \mathcal{J}_{\mathbf{R}} \tag{21}$$

where $\mathcal{J}_{\mathbf{R}}$ is the ideal generated by $\mathbf{R} \subset \mathcal{L}ie_{\mathbf{k}}\langle X\rangle$.

42. When $\mathbf{R}$ has a lot of "commutation generators" i.e. polynomials like $[x, y]$ with $x, y \in X$, it is more convenient to compute with a "reduced category of generators" like that of alphabets with commutations rather than alphabets (sets).

[a] Among many ways like: matrices, vector fields, derivations, &c.

Relations with generators/4

Free partially commutative structures.

43. There is a set of structures (free partially commutative, see [11]).
44. For monoids, given $\theta \subset X \times X$ is a reflexive undirected graph (i.e. $\Delta_X \subset \theta \subset X \times X$ where X is the alphabet or set of generators), one has
$$M(X,\theta) = \langle X; (xy = yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}} \tag{22}$$
45. These structures are compatible with Lazard's elimination and MRS factorization. This can proved using $\mathbf{k}[M(X,\theta)] = \mathcal{U}(Lie_{\mathbf{k}}(X,\theta))$.
46. A unipotent Magnus group with a nice Log-Exp correspondence can be defined more generally for every locally finite monoid. Is there a general MRS factorization ?
47. In the sound cases, what is the combinatorics of different orders ? (Not increasing or decreasing Lyndon words.).

48. We recall here the mechanism of adjunction w.r.t. a functor. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories and $F : \mathcal{C}_2 \to \mathcal{C}_1$ a (covariant) functor between them

$$\begin{tikzcd}
\mathcal{C}_1 & \mathcal{C}_2 \arrow[l, dotted, "F_{12}"'] \\
U \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_U"'] & V \\
& G_{21}(U) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

49. Here, we will consider $\mathcal{C}_1 = \mathbf{CSet}$ and $\mathcal{C}_2 = \mathbf{k} - \mathbf{Lie}$.
50. We define **CSet** as the category such that $Ob(\mathbf{CSet})$ is the class of reflexive undirected graphs (i.e. such that, for some X, $\Delta_X \subset \theta \subset X \times X$) and a morphism $\theta_1 \to \theta_2$ is a map $f : X_1 \to X_2$ such that "f respects the commutations" i.e.
$$(x, y) \in \theta_1 \Longrightarrow (f(x), f(y)) \in \theta_2$$

Relations with generators/6

51 Now, for $\mathfrak{g} \in \mathbf{k} - \mathbf{Lie}$,

$$F_{12}(\mathfrak{g}) = \{(u,v) \in \mathfrak{g}^2 \mid [u,v] = 0\} \tag{23}$$

one can verify that $F_{12}(\mathfrak{g}) \in Ob(\mathbf{CSet})$ i.e. that it is a reflexive undirected graph with set of vertices $\mathfrak{g}$.

52 Conversely, a reflexive undirected graph $\theta \subset X^2$ being given, the free object w.r.t. F_{12} is the free partially commutative Lie algebra

$$\mathcal{L}ie_{\mathbf{k}}\langle X, \theta\rangle := \langle X; ([x,y])_{(x,y)\in\theta}\rangle_{\mathbf{k}-\mathbf{Lie}} \tag{24}$$

53 A word $w \in X^*$ being given, we define ad_w in the usual way and show that $ad_\bullet : \ X^* \mapsto End(\mathcal{L}ie_{\mathbf{k}}\langle X, \theta\rangle)$ respects the commutations of θ and that it is in fact a map $M(X,\theta) \to End(\mathcal{L}ie_{\mathbf{k}}\langle X, \theta\rangle)$.

Relations with generators/7

Lazard's eliminiation (general).

54. Suppose that you have partitioned your alphabet X as $X = Z + B$ $(= Z \sqcup B)$ such that Z be "totally noncommutative" i.e. nobody commutes with nobody. In other words

$$(x,y) \in Z^2 \Longrightarrow (x,y) \notin \theta \tag{25}$$

Then

1. The Lie ideal $\mathcal{J} = (\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle)_{\,|\;|_Z \geq 1}$ is a Free Lie algebra (without commutations) and alphabet

$$C_Z(B) = \{ad_w(z)\}_{w \in M(B,\theta_B) z \in Z,\ AT(wz)=\{z\}} \tag{26}$$

and

$$\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle \simeq \mathcal{L}ie_{\mathbf{k}}\langle C_Z(B)\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B,\theta_B\rangle \tag{27}$$

Relations with generators/8

An example

55 This is the case of the Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n}$ presented with the set of generators $\mathcal{T}_n^{<} = \{t_{i,j}\}_{1\leq i<j\leq n}$ and relations

$$\mathbf{R_2} = \begin{cases} [t_{i,j}, t_{i,k} + t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ [t_{i,j} + t_{i,k}, t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ [t_{i,j}, t_{k,l}] & \text{for } \begin{matrix} 1 \leq i < j \leq n, \\ 1 \leq k < l \leq n, \end{matrix} \text{ and } |\{i,j,k,l\}| = 4. \end{cases} \tag{28}$$

56 As n increases, one observes that there are more and more commutation relations (i.e. the relator becomes sparse).

57 This is the reason why we will present Lazard's elimination with the category of sets with commutation rather than that of sets.

Next time

58. Let us take again the notion of $\mathbb{B}$-graded set and consider

$$
\begin{tikzcd}
\mathcal{C}_1 & & \mathcal{C}_2 \arrow[ll, dotted, "F_{12}"'] \\
U \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_U"'] & & V \\
& & G_{21}(U) \arrow[u, "\hat{f}"']
\end{tikzcd}
$$

Where $\mathcal{C}_2$ is the category of $\mathbb{B}$-graded **k**-Lie algebras, $\mathcal{C}_1$ is the category of $\mathbb{B}$-graded sets and F_{12} the functor such that $F_{12}(\mathfrak{g}) = F_{12}(\mathfrak{h} \rtimes F_{12}(\mathfrak{b})) = S$ such that $S_1 = \mathfrak{h}$ and $S_0 = \mathfrak{b}$.

59. We will present LET as a solution of this universal problem.

Concluding remarks

60. We have seen semi-direct products of Lie algebras as a universal problem (see CCRT[30] for details).
61. Many presentations considered in combinatorial group theory and combinatorial Lie algebra theory (in particular arising from topology and graph theory) have a lot of commutations.
62. The natural structure to compute with them is to use presentation with "generators and relations".
63. We have seen the general Lazard's elimitation for these structures and the category of Lie algebras.
64. This Lazard elimitation generalizes the classical one and provides a semi-direct product.
65. There is a map from this semi-direct product to any semi-direct product of Lie algebras.
66. Detailed implementation of this will be the subject of a forthcoming CCRT.

Thank you for your attention.

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